Modal Analysis of Constrained Multibody Systems Undergoing Constant Accelerated Motions

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The modal characteristics of constrained multibody systems undergoing constant accelerated motions are investigated in this paper. Relative coordinates are employed to derive the equations of motion, which are generally nonlinear in terms of the coordinates. The dynamic equilibrium position of a constrained multibody system needs to be obtained from the nonlinear equations of motion, which are then linearized at the dynamic equilibrium position. The mass and the stiffness matrices for the modal analysis can be obtained from the linearized equations of motion. To verify the effectiveness and the accuracy of the proposed method, two numerical examples are solved and the results obtained by using the proposed method are compared with those obtained by analytical and other numerical methods. The proposed method is found to be accurate as well as effective in predicting the modal characteristics of constrained multibody systems undergoing constant accelerated motions.

Key Words : Dynamic Equilibrium, Constant Accelerated Motion, Constrained Multibody System, Modal Analysis, Relative Coordinate

1. Introduction

Mechanical systems can be modeled as constrained multibody systems that consist of rigid and flexible bodies, joints, springs, dampers, forces and so on. In general, the equations of motion governing constrained multibody systems consist of nonlinear differential and algebraic equations. To obtain the response of a constrained multibody system, several computational methods (Sheth et al., 1972 ; Orlandea et al., 1977 ; Paul, 1977 ; Haug et al., 1982) have been introduced since early 1960's. Several commercial programs for multibody system analysis (for instance, ADAMS, DADS, and RecurDyn) are available nowadays. By using these programs, kinematic, dynamic, and static equilibrium analyses

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of constrained multibody systems can be performed. If a constrained multibody system has a static equilibrium position, its modal characteristics, which are often important for system design, can be also obtained. Equations of motion are linearized at the static equilibrium position and the mass and the stiffness matrices for the modal analysis can be obtained. Sohoni and Whitesell (1986) introduced a linearization method based on a generalized coordinate partitioning method in which dependent coordinates are eliminated. Lynch and Vanderploeg (1995) proposed another linearization method employing QR decomposition by which a constrained set of equations can be converted to an unconstrained set of equations. By using these methods, the modal characteristics of a constrained multibody system in state of rest could be obtained.

There exists a state of dynamic equilibrium which resembles static equilibrium. In the state of dynamic equilibrium, a part of the generalized coordinates have constant values even though the system varies with time. Therefore, in the state of motion, one may choose a set of generalized

coordinates which become constant. This state which is determined by the set of generalized coordinates will be hereinafter called a dynamic equilibrium state. The dynamic equilibrium state of the system can be easily calculated if a proper generalized coordinate is chosen. Therefore it is important to choose a proper set of generalized coordinates if one has the purpose to find the dynamic equilibrium state effectively. Relative angles and displacements between bodies are the best candidates for the purpose. Such coordinates are often called relative coordinates. Bae and Haug (1987) introduced a multibody formulation by employing relative coordinates. However, they did not provide a method either to find the dynamic equilibrium state or to calculate the modal characteristics of a constrained multibody system.

Constrained multibody systems undergoing constant accelerated motions exhibit distinct modal characteristics. As their accelerations vary, their natural frequencies usually vary, too. The varying modal characteristics need to be predicted accurately for a proper system design. However, to the best of the authors' knowledge, the varying modal characteristics cannot be calculated directly by using any existing multibody analysis programs (though some of them have the capability to calculate the modal characteristics of constrained multibody systems in states of rest). Actually, even dynamic equilibrium states cannot be calculated efficiently by using any existing commercial codes. To obtain a dynamic equilibrium state of a constrained multibody system (by using existing commercial codes), a transient dynamic analysis should be performed with a prescribed motion, which increases smoothly and reaches constant acceleration. Then the modal characteristics can be obtained by analyzing the oscillatory motion around the dynamic equilibrium state. If the system has one degree of freedom, one may count the number of oscillation to find the natural frequency. However, if the system has more than one degree of freedom, the oscillatory motion has to be analyzed by using a Fourier transformation method. This procedure is time consuming and obviously not proper for

design.

The purpose of this paper is to propose a numerical method to calculate the modal characteristics of constrained multibody systems undergoing constant accelerated motions. Relative coordinates are employed to describe a constrained multibody system and a velocity transformation matrix is employed to derive the equations of motion. If the system has closed kinematic loops, constraint forces arising from the closed loops can be eliminated by using the velocity transformation matrix. A formulation to seek the dynamic equilibrium state of a constrained multibody system undergoing constant accelerated motions is first presented. Then linearization procedures for open and closed loop systems are presented. To verify the effectiveness and the accuracy of the proposed method, numerical examples are solved and the results are compared with those obtained by other methods.

2. Equations of Motion

In 3 dimensional space, a free rigid body's configuration can be determined by six coordinates. Three scalar variables are employed to determine the position of a point (for instance, the center of mass) fixed in the rigid body and three successive rotation angles (often named as Euler angles) are employed to determine the orientation of the body. The coordinate set of the i-th body of a multibody system is denoted as x_i. If a multibody system consists of n rigid bodies, its total coordinate set (named and denoted as a Cartesian coordinate set x) consists of n coordinate sets as follows:

$$x = [x_1^T \ x_2^T \ \cdots \ x_n^T]^T \qquad (1)$$

By employing the Cartesian coordinate set, the equations of motion of a constrained multibody system can be derived (Nikravesh, 1988) as follows:

$$M\ddot{x} + \Phi_x^T \lambda = Q \qquad (2)$$

where M is a mass matrix, Q is a generalized force matrix, and λ is a Lagrange multiplier

matrix. The matrix $\boldsymbol{\Phi}$ represents algebraic constraint equations that originate from kinematic joints and $\boldsymbol{\Phi}_x$ is the Jacobian matrix which is the partial derivative of the constraint equations with respect to the Cartesian coordinate set.

A closed loop multibody system can be transformed into a open loop multibody system by cutting joints as shown in Fig. 1. The number of cut joints is same as the number of closed loops. The constraint equations that originate from the cut joints are denoted as $\boldsymbol{\Phi}^c$ and the rest of the constraint equations are denoted as $\boldsymbol{\Phi}^r$. So, the total constraint equations consists of the two sets of equations as follows :

$$\boldsymbol{\Phi}=[\boldsymbol{\Phi}^{cT} \quad \boldsymbol{\Phi}^{rT}]^T \qquad (3)$$

Now Eq. (2) can be rewritten as follows :

$$M\ddot{x}+\boldsymbol{\Phi}_x{}^{cT}\lambda^c+\boldsymbol{\Phi}_x{}^{rT}\lambda^r=Q \qquad (4)$$

where λ^c and λ^r represent the Lagrange multipliers for $\boldsymbol{\Phi}^c$ and $\boldsymbol{\Phi}^r$, respectively.

The equations of motion can be transformed into a reduced form by employing relative coordinates. For the purpose, the following relation is often employed.

$$\dot{x}=B\dot{q} \qquad (5)$$

where $\dot{q}$ is the time derivative of relative coordinates q and the transpose of B is the null space of $\boldsymbol{\Phi}_x^r$. This relation is often called the velocity transformation (Kim and Vanderploeg, 1986). One may choose some of q (which will
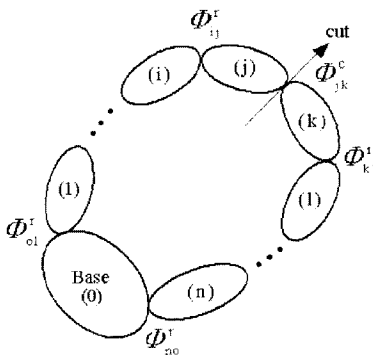
be denoted as q_P) to prescribe a constant acceleration for a constrained multibody system. The rest of q will be denoted as q_R. Then, Eq. (5) can be rewritten as follows :

$$\dot{x}=B_P\dot{q}_P+B_R\dot{q}_R \qquad (6)$$

Now, by differentiating Eq. (6), the following equations can be obtained.

$$\ddot{x}=B_P\ddot{q}_P+B_R\ddot{q}_R+\dot{B}_P\dot{q}_P+\dot{B}_R\dot{q}_R \qquad (7)$$

Now substituting Eq. (7) into Eq. (4) and pre-multiplying the results by B_R^T, one obtains the following equation.

$$B_R^T[M(B_P\ddot{q}_P+B_R\ddot{q}_R+\dot{B}_P\dot{q}_P+\dot{B}_R\dot{q}_R) \\ +\boldsymbol{\Phi}_x{}^{cT}\lambda^c+\boldsymbol{\Phi}_x{}^{rT}\lambda^R]=B_R^TQ \qquad (8)$$

Note that $B_R^T\boldsymbol{\Phi}_x{}^{rT}$ is the null matrix since is the null space B_R^T of $\boldsymbol{\Phi}_x{}^{rT}$. Now the following relation can be used to further simplify the above equation.

$$\boldsymbol{\Phi}_{q_R}=\frac{\partial\boldsymbol{\Phi}}{\partial x}\frac{\partial x}{\partial q_R}=\boldsymbol{\Phi}_x\frac{\partial\dot{x}}{\partial\dot{q}_R}=\boldsymbol{\Phi}_xB_R \qquad (9)$$

where the dot cancellation law (Rosenberg, 1977) is employed. By using Eq. (9), Eq. (8) can be rewritten as follows :

$$M^*\ddot{q}_R+\boldsymbol{\Phi}_{q_R}^{cT}\lambda^c=Q^* \qquad (10)$$

where

$$M^*=B_R^TMB_R \qquad (11)$$

$$Q^*=B_R^TQ-B_R^T(MB_P\ddot{q}_P+M\dot{B}_P\dot{q}_P+M\dot{B}_R\dot{q}_R) \qquad (12)$$

The acceleration constraint equations, the second time derivatives of the constraint equations $\boldsymbol{\Phi}^c=0$, can be written as follows :

$$\boldsymbol{\Phi}_{q_R}^c\ddot{q}_R=\gamma^c \qquad (13)$$

where

$$\gamma^c=-(\boldsymbol{\Phi}_{q_R}^c\dot{q}_R)_{q_R}\dot{q}_R-2\boldsymbol{\Phi}_{q_Rt}^c\dot{q}_R-\boldsymbol{\Phi}_{tt}^c \qquad (14)$$

Equations (10) and (13) are used to perform a dynamic analysis of a constrained multibody system undergoing a constant accelerated motions.

3. Linearization and Modal Equation

In order to find the modal characteristics of a

Fig. 1 Schematic representation of a closed loop system

constrained multibody system undergoing constant acceleration motions, the dynamic equilibrium state of the system has to be found first. At the dynamic equilibrium state, $\dot{q}_R$ and $\ddot{q}_R$ become zero. Substituting these relations into the equations of motion (10), one obtains the following equation.

$$B_R^T[MB_P\ddot{q}_P+M\dot{B}_P\dot{q}_P-Q]+\Phi_{q_R}^{cT}\lambda^c=0 \quad (15)$$

Since q_P are chosen for translational motion, the sub-matrix B_P should be constant matrix. Therefore the time derivative of B_P becomes zero. So, the following algebraic equations are obtained to find the equilibrium state.

$$B_R^T[MB_P\ddot{q}_P-Q]+\Phi_{q_R}^{cT}\lambda^c=0 \quad (16)$$

The above equations along with the constraint equations ($\Phi^c=0$) have to be solved to find the dynamic equilibrium state. Since these equations are nonlinear, the well-known Newton-Raphson procedure can be used to solve them. By solving the equations, q_R and λ^c can be obtained. The values of q_R which are obtained from the equilibrium equations will be used later to obtain the modal equations.

To obtain the modal equations, Eq. (10) has to be transformed into a minimum set of equations of motion. For the purpose, q_R should be partitioned as follows:

$$q_R=[u^T \ v^T]^T \quad (17)$$

where u and v represent dependent and independent coordinate sets, respectively. Several methods (Wehage and Haug, 1982; Nikravesh and Srinivasan, 1985) of selecting independent coordinate sets are known. Now, $\dot{q}_R$ can be expressed as a function of the independent velocity vector $\dot{v}$ as follows:

$$\dot{q}_R=R\dot{v} \quad (18)$$

where R is defined as follows:

$$R=\begin{bmatrix} \Phi_u^{c-1}\Phi_v^c \\ I \end{bmatrix} \quad (19)$$

Pre-multiplying Eq. (10) by R^T results in a minimum set of equations of motion as follows:

$$R^TM^*R\dot{v}+R^TM^*\dot{R}\dot{v}-R^TQ^*=0 \quad (20)$$

Note that R^T is the null space of $\Phi_{q_R}^{cT}$. Equation (20) can be linearized at the dynamic equilibrium state q_R^* and the following modal equations can be obtained to investigate the modal characteristics of the system.

$$\hat{M}^*\delta\ddot{v}+\hat{C}^*\delta\dot{v}+\hat{K}^*\delta v=0 \quad (21)$$

where $\hat{M}^*$, $\hat{C}^*$ and $\hat{K}^*$ are the linearized mass, damping and stiffness matrices of the modal equations and they can be calculated at the dynamic equilibrium state q_R^* as

$$\hat{M}^*=R^TM^*R \quad (22)$$

$$\hat{C}^*=R^TM^*\dot{R}-\frac{\partial}{\partial\dot{v}}(R^TQ^*) \quad (23)$$

$$\hat{K}^*=\frac{\partial}{\partial v}(-R^TQ^*) \quad (24)$$

The simple finite difference method is employed to obtain them in the present study. For instance, the following equation represents the simple finite difference method to calculate $\hat{K}^*$:

$$\hat{K}^*=\frac{h(v^*+\delta v)-h(v^*)}{\delta v} \quad (25)$$

where h denotes $-R^TQ^*$ in Eq. (20) and v^* represents the independent coordinate value in the dynamic equilibrium position q_R^*. Note that Eq. (21) is a homogeneous equation. Non-homogeneous terms are not needed to analyze the free vibration modal characteristics.

4. Numerical Results and Discussion

1 DOF (degree of freedom) swing pendulum system attached to a moving base which undergoes a constant accelerated motion is shown in Fig. 2. This example has an open kinematic loop. The uniform bar, which has mass $m=10$ kg and length $L=1$ m, is connected by a revolute joint. The moving base is connected to the ground by a translational joint. Friction force between the moving base and ground is not considered in this analysis. If the moving base is made to move with constant acceleration, the relative angle θ between the vertical axis and the pendulum remains

constant. Therefore the relative angle θ is chosen as the generalized coordinate. The analytical equation of motion is derived as follows :

$$\frac{1}{3} mL^2 \ddot{\theta} + \frac{1}{2} mLa \cos \theta + \frac{1}{2} mgL \sin \theta = 0 \quad (26)$$

where a is the constant acceleration value. At dynamic equilibrium state, $\ddot{\theta}$ becomes zero. Therefore, the following equilibrium equations are obtained.

$$\frac{1}{2} mLa \cos \theta + \frac{1}{2} mgL \sin \theta = 0 \qquad (27)$$

This equilibrium equation is the same as the equation which is obtained by using the proposed method. The detailed derivation procedure for the proposed method is given in Appendix A. Using the proposed method, the linearized mass and stiffness matrices are calculated as follows :
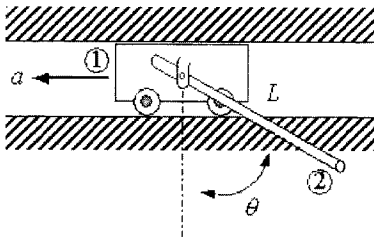
$$\hat{M}^* = \frac{1}{3} mL^2 \qquad (28)$$

Fig. 2   Swing pendulum attached to a moving base

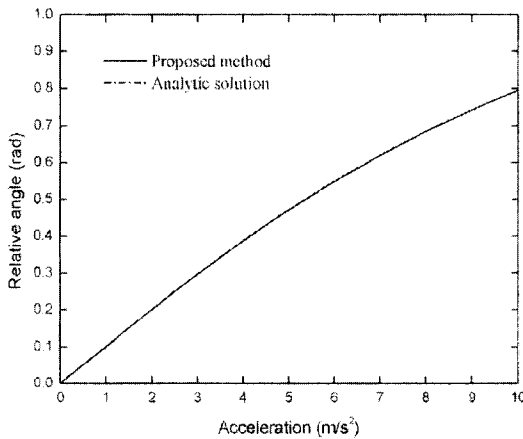Fig. 3   Variation of the dynamic equilibrium position

$$\hat{K}^* = -\frac{1}{2} mLa \sin \theta + \frac{1}{2} mgL \cos \theta = 0 \quad (29)$$

Comparing Eq. (26) to Eqs. (28) and (29), one can find that two linearized equations are identical. Thus, the two numerical results for the equilibrium position and natural frequency should be identical, too. Figure 3 shows the variation of θ versus the acceleration of the moving base. The variation of the natural frequency versus the acceleration of the moving base is shown in Fig. 4.
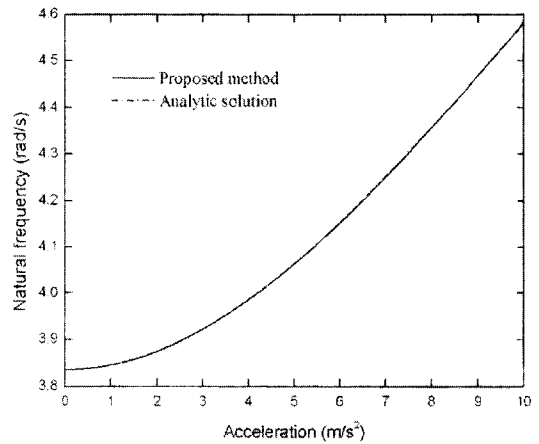
Figure 5 shows a closed loop mechanism which

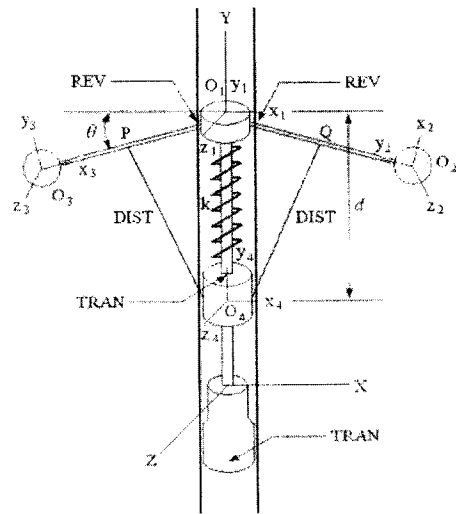Fig. 4   Natural frequency variation versus acceleration

Fig. 5   A closed loop system undergoing constant acceleration

has two closed kinematic loops. Body 1 of the system is the moving base which is driven toward $-Y$ direction by a constant acceleration ($\ddot{y}_1 = -10.0$ m/s^2); body 2 and body 3 are pendulums which have a sphere mass at each end; and body 4 is the collar. The moving base is connected to the ground by a translational joint. The moving base and the pendulums are connected by revolute joints; the moving base and the collar are connected by a translational joint and a spring; and the collar and the pendulums are connected by distance joints having fixed distance of 0.1092 m. The stiffness and the free length of the spring are 1000 N/m and 0.15 m, respectively.

Table 1 shows the inertia properties of the constituting bodies and Table 2 shows the coordinates of some points (shown in Fig. 5) that determine the configuration of the system. Figure 6 shows the topology of the system. Since this system has two closed loops, two distance joints should be cut.

From the equilibrium analysis, Fig. 7 shows the relative distance d between body 1 and body 4. At dynamic equilibrium state, two results are almost identical. To simulate this analysis using a commercial program, however, the body 1 should be accelerated smoothly and a practical damping

($c_t = 30$ Nsec/m) should be imposed to obtain the equilibrium position. Note that time integration is required to obtain the results. However,
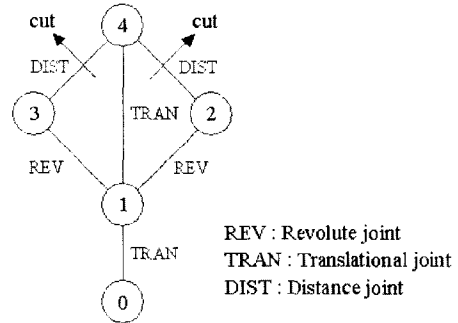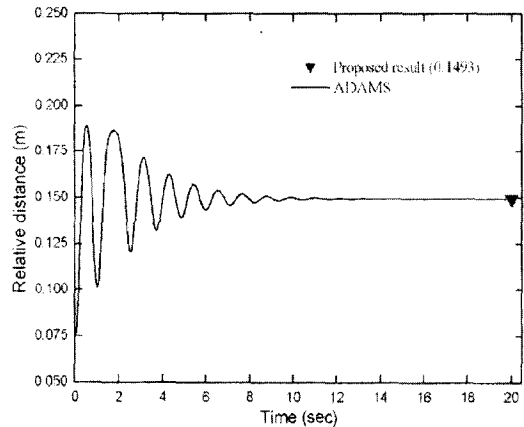

Fig. 6   Tree structure of the system

Fig. 7   Dynamic equilibrium position of the system

Table 1   Inertia properties of the constituting bodies

Body	Mass [kg]	Moment of inertia [kg·m²]		
		$I_{x'x'}$	$I_{y'y'}$	$I_{z'z'}$
Body 1	200.0	25.0	50.0	25.0
Body 2	1.0	0.1	0.1	0.1
Body 3	1.0	0.1	0.1	0.1
Body 4	1.0	0.15	0.125	0.15

Table 2   Initial position of points shown in Fig. 5

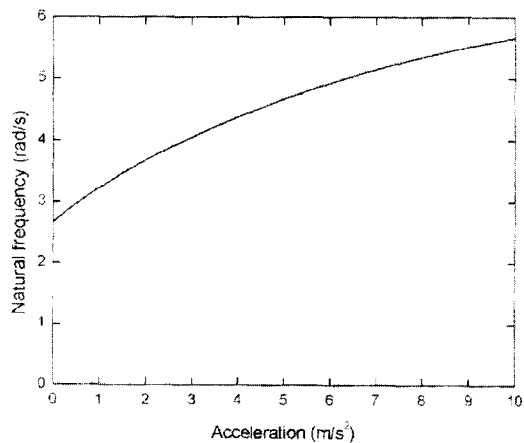| Point	Initial Position [m]
O_1	[0.0, 0.2, 0.0]
O_2	[−0.16, 0.2, 0.0]
O_3	[0.16, 0.2, 0.0]
O_4	[0.0, 0.1256, 0.0]
P	[−0.08, 0.2, 0.0]
Q	[0.08, 0.2, 0.0]

Fig. 8   Natural frequency variation versus acceleration

using the proposed method, the dynamic equilibrium position could be obtained without time integration. The variation of the natural frequency versus the acceleration which is obtained by using the proposed method is also shown in Fig. 8.

5. Conclusions

In this paper, a computational algorithm is proposed to find the modal characteristics of multibody systems undergoing constant accelerated motions. Such multibody systems are often found in engineering examples like launching rockets and missiles. The equations of motion are derived by employing relative coordinates and linearized at the dynamic equilibrium position. The mass and the stiffness matrices for the modal analysis can be obtained from the linearized equations. To verify the effectiveness and the accuracy of the proposed method, two numerical examples are solved. The results obtained by using the proposed method are compared to those obtained by analytical methods. It is proved that the proposed method provides accurate modal characteristics of multibody systems undergoing constant accelerated motions. The proposed method can be easily implemented into any existing multibody analysis programs. Since the method does not necessitate numerical integration, it is superior to any existing methods that employ numerical integration.

Acknowledgment

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Appendix A
Derivation of Equation

The equilibrium equation of the open loop system undergoing constant accelerated motion is as follows :

$$B_R^T(MB_P\ddot{q}_P - Q) = 0 \qquad (a1)$$

where $\ddot{q}_P$ is given as a constant value a and B_P, B_R, and Q are given as

$$B_P = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B_R = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{L}{2}\cos\theta \\ \dfrac{L}{2}\sin\theta \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -mg \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (a2)$$

Now substituting Eq. (a2) into Eq. (a1), one can obtain the following equation.

$$\frac{1}{2}mLa\cos\theta + \frac{1}{2}mgL\sin\theta = 0 \qquad (a3)$$

This result is identical to Eq. (27) which is obtained by analytical method.